COLLAPSE OF RANDOM TRIANGULAR GROUPS: A CLOSER LOOK

SYLWIA ANTONIUK, TOMASZ ŁUCZAK, AND JACEK ŚWIĄTKOWSKI

Abstract. The random triangular group $\Gamma(n,t)$ is a group given by a presentation $P = \langle S|R\rangle$, where S is a set of n generators and R is a random set of t cyclically reduced words of length three. The asymptotic behavior of $\Gamma(n,t)$ is in some respects similar to that of widely studied density random group introduced by Gromov. In particular, it is known that if $t \le n^{3/2-\varepsilon}$ for some $\varepsilon > 0$, then with probability $1-o(1)$ $\Gamma(n,t)$ is infinite and hyperbolic, while for $t \ge n^{3/2+\varepsilon}$, with probability $1-o(1)$ it is trivial. In this note we show that $\Gamma(n,t)$ collapses provided only that $t \ge Cn^{3/2}$ for some constant $C > 0$.

By $\langle S|R\rangle$ we denote a presentation P of a group, where S is the set of generators and R the set of relations. A presentation P is a *triangular presentation* if the set R of relations consists of distinct cyclically reduced words of length three only (over the alphabet $S \cup S^{-1}$ which consists of generators and their formal inverses). A *triangular group* is a group given by a triangular presentation.

In the paper we investigate the properties of groups given by presentations $P = \langle S|R\rangle$, where S is a set of n generators and R is a random subset of the set $\mathcal{T}$ of all $N = 2n(4n^2 - 6n + 3) \sim 8n^3$ distinct cyclically reduced words of length three, that is words of the form abc, where $a \neq b^{-1}$, $b \neq c^{-1}$ and $c \neq a^{-1}$. Here we consider two models of a random triangular group. In the random uniform model $\Gamma(n,t)$, the set of relations R is chosen uniformly at random from the family of all $\binom{N}{t}$ subsets of $\mathcal{T}$ of size t. In the binomial model $\Gamma(n,p)$, we select each element from $\mathcal{T}$ independently with probability p. We shall study the asymptotic properties of these two models as $n \to \infty$. In particular, we say that for a given function $t(n)$ [or $p(n)$] the group $\Gamma(n,t(n))$ [resp. $\Gamma(n,p(n))$] has a property a.a.s. (asymptotically almost surely) if the probability that the random group has this property tends to 1 as $n \to \infty$. It should be mentioned that, as is well known in the theory of random structures, the asymptotic behavior of $\Gamma(n,t)$ and $\Gamma(n,p)$ can

---

Date: January 14, 2013.

2010 *Mathematics Subject Classification.* Primary: 20P05; secondary: 20F05, 05C80.

Key words and phrases. random group, collapse, intersection graphs.

Jacek Świątkowski partially supported by the Polish Ministry of Science and Higher Education (MNiSW), Grant N201 541738.

be proved to be very similar provided $t \sim Np$. We remark that $\Gamma(n,t)$ is basically identical with the model of random group introduced by Żuk [9] who also pointed out the connection between his model and a better known construction of random groups introduced by Gromov [4] (a more precise explanation of this relationship is given in [7] and [6]).

It is known (see [9] and Theorem 29 in [7]) that when the density $d = \frac{\ln t}{3 \ln n}$ of $\Gamma(n,t)$ is smaller than $1/2 - \varepsilon$ for some $\varepsilon > 0$, then a.a.s. $\Gamma(n,t)$ is infinite and hyperbolic, while for $d > 1/2 + \varepsilon$ a.a.s. $\Gamma(n,t)$ collapses to the trivial group. Here we study more precisely the case when the density is $1/2 + o(1)$. As it was pointed out in [7] part IV.a, this is the usual question after a talk on random groups. The main results of this note can be stated as follows.

Theorem 1. *There exists an absolute constant $C_1 > 0$ such that if $p \geq C_1 n^{-3/2}$ then a.a.s. $\Gamma(n,p)$ is trivial.*

Theorem 2. *There exists an absolute constant $C_2 > 0$ such that if $t \geq C_2 n^{3/2}$ then a.a.s. $\Gamma(n,t)$ is trivial.*

Our proof of Theorem 1 is based on the following fact concerning so called random intersection graphs. An *intersection graph* is a graph on vertex set V where each vertex v is assigned a set of features W_v which is a subset of a ground set of features W. Two vertices v_1 and v_2 are adjacent if they share a common feature, i.e. if $W_{v_1} \cap W_{v_2} \neq \emptyset$. In a *random intersection graph* $G_{n,m,\rho}$ the set of vertices V is of size n, the set of features W is of size m and for any vertex v we assign each feature from W to v independently with probability ρ.

Note that the probability that two vertices of $G_{n,m,\rho}$ are adjacent is roughly

$$\hat{p} = 1 - (1 - \rho^2)^m \sim \rho^2 m.$$

If we denote $m = n^\alpha$ then it turns out that for $\alpha > 1$ the random intersection graph $G_{n,m,\rho}$ behaves in a similar manner as the random graph $G(n, \hat{p})$, in which each pair of vertices is adjacent with probability $\hat{p}$, independently for each pair. In 1960 Erdős and Rényi [3] in their seminal paper on the evolution of random graphs proved that when $n\hat{p} > c > 1$, then a.a.s. in $G(n,\hat{p})$ one can find a unique giant component which contains a positive fraction of all vertices (more precisely, they showed it for an equivalent uniform model of the random graph, for details see [3] or [5]). Berisch [1] showed that the fact that in $G_{n,m,\rho}$ the edges do not occur independently does not affect very much the emergence of the giant component for the range of parameters we are interested in. In particular, a special case of Theorem 1 in [1] is the following.

Lemma. *Let $G_{n,m,\rho}$ be a random intersection graph with $m = n^\alpha$, where $\alpha > 1$, and $\rho^2 m = \frac{\beta}{n}$. Furthermore, for $\beta > 1$, let γ be the*

unique solution to $\gamma = \exp(\beta(\gamma-1))$ in the interval $(0,1)$. Then a.a.s. the size of the largest component in $G_{n,m,\rho}$ is of order $(1+o(1))(1-\gamma)n$.

In particular, if $\beta \geq 1.42$ then a.a.s. $G_{n,m,\rho}$ contains a component of size at least $0.52n$.

Proof of Theorem 1. We shall show that the assertion holds whenever $C_1 \geq 1.2$. Let us partition the set of generators S into two sets S_1 and S_2, where $|S_1| = \lceil |S|/2 \rceil = \lceil n/2 \rceil$. We generate the set of relations R of $\Gamma(n,p)$ in two stages. Firstly, we consider the relations which contain exactly one element from $S_1 \cup S_1^{-1}$, and select each such relations independently with probability $p = 1.2n^{-3/2}$. Then, we select each of the remaining candidates for relations with the same probability $p = 1.2n^{-3/2}$. We denote by R_1 the random set of relations generated in the first stage, and by R_2 the relations chosen in the second stage, so that $R = R_1 \cup R_2$.

Consider an auxiliary random intersection graph $G_{n,m,\rho}$ with vertex set $V = S_1 \cup S_1^{-1}$, the set of objects $W = \{cd : c, d \in S_2 \cup S_2^{-1}, c \neq d^{-1}\}$ and such that for any $a \in V$ a feature $cd \in W$ is assigned to a in $G_{n,m,\rho}$ if we have generated at least one relation from the set $\{acd, cda, dac\}$. Note that here $m = n(n-1)$ and $\rho = 1-(1-p)^3 \geq p$. Therefore, $\beta = \rho^2 mn \geq p^2 n^2 (n-1) \geq 1.44\frac{n-1}{n}$.

Using Lemma we infer that a.a.s. $G_{n,m,\rho}$ contains a large component L of at least $0.52n$ vertices. Note however, that if two vertices a, b are adjacent in L, then they share a common feature $cd \in W$, which implies that $a = d^{-1}c^{-1} = b$ in $\Gamma(n,p)$. Moreover, since $|L| > |S_1|$, for some $s \in S_1$, L must contain both s and s^{-1}. Consequently all elements of L are not only equal to s, but also satisfy the condition $s^2 = e$.

Now consider relations generated in the second stage. Let X count the number of elements s from L such that for any $s', s'' \in L$ the relation $ss's''$ does not belong to R_2. Then

$$\Pr(X > 0) \leq \mathbb{E}X \leq 0.52n(1-p)^{0.25n^2} \leq 0.52n\exp(-0.36\sqrt{n}),$$

and tends to 0 as $n \to \infty$. Hence, a.a.s. L contains three elements s, s', s'' such that $ss's'' \in R_2$. However, since all elements from L are equal, we conclude that $s^3 = e$. This, together with the condition that $s^2 = e$, implies that a.a.s. all generators or inverses of generators contained in L are equivalent to the identity.

In a similar way one can estimate the number of generators s not contained in L, for which no triple $ss's''$ belongs to R_2, where $s', s'' \in L$. Again, the probability that there is at least one such generator tends to 0 as $n \to \infty$, therefore a.a.s. each generator from S is equivalent to the identity and the assertion follows. $\square$

Proof of Theorem 2. Note that the property that $\langle S|R\rangle$ is trivial is a monotone property of the set of relations R. Therefore, to show that the asymptotic behavior of $\Gamma(n,t)$ in terms of generating a trivial

group is similar to the asymptotic behavior of $\Gamma(n,p)$, one can use exactly the same method as in the proof of asymptotic equivalence for monotone properties of the uniform random graph model $G(n,M)$ and the binomial random graph model $G(n,p)$ (see [5]). $\square$

The result proved above together with the following conjecture (stated for $\Gamma(n,t)$, but having the analogue for $\Gamma(n,p)$), provide a sharp threshold for collapsibility of a triangular random group.

Conjecture. *There exists a constant $c > 0$ such that for every $t \leq cn^{3/2}$ a.a.s. $\Gamma(n,t)$ is infinite.*

As a matter of fact, we conjecture that for small c and for $t < cn^{3/2}$ a.a.s. $\Gamma = \Gamma(n,t)$ is an aspherical group, i.e. it has no reduced spherical van Kampen diagram. In this case the presentation complex C of this group is aspherical, and hence it is a classifying space for this group, which has the following consequences:

- the Euler characteristic $\chi(\Gamma) = \chi(C) = 1 - n + t$;
- Γ is torsion-free (see [2], p. 187).

Since a group Γ with $\chi(\Gamma) \neq 1$ is nontrivial (and this is true when $t \sim n^{3/2}$), the above "asphericity conjecture" implies Conjecture (for $t \ll n^{3/2}$ Conjecture is true by the earlier mentioned result of Żuk).

As a circumstantial evidence supporting the above "asphericity conjecture" we remark that it is not hard to show that for small c and $t \leq cn^{3/2}$ a.a.s. $\Gamma(n,t)$ has no reduced spherical van Kampen diagram with cells corresponding to pairwise distinct relations. Moreover, for every constant A, a.a.s. $\Gamma(n,t)$ has no reduced spherical van Kampen diagram with fewer than A cells.

Finally, let us mention that one can easily use the same argument to show that for every ℓ there exists a constant a_ℓ such that a.a.s. as $m \to \infty$ the random group given by presentations $P = \langle S|R\rangle$, where S is a generating set of cardinality m, and where R is a random subset of the set of cyclically reduced words of length ℓ, collapses to the trivial group provided only $|R| \geq a_\ell(2m-1)^{\ell/2}$. If ℓ is a prime one can just mimic our argument; in the remaining cases there is a tiny technical problem to deal with rare cyclical words which can be easily solved. This result matches nicely a theorem of Gady Kozma, mentioned in [8], which concerns the Gromov model when m is fixed and $\ell \to \infty$.

Acknowledgment

We would like to thank Katarzyna Rybarczyk for her valuable remarks on random intersection graphs.

References

[1] M. Behrisch, *Component evolution in random intersection graphs*, Electr. J. Comb. 14(1) (2007) R17.

[2] K. Brown, "*Cohomology of groups*", Graduate Texts in Mathematics 87, Springer, 1982.

[3] P. Erdős, A. Rényi, *On the evolution of random graphs*, Publ. Math. Inst. Hung. Acad. Sci., Ser. A 5 (1960), 17–61.

[4] M. Gromov, *Asymptotic invariants of infinite groups. Geometric Group Theory*, London Math. Soc. Lecture Note Ser. 182 (1993), 1–295.

[5] S. Janson, T. Łuczak, and A. Ruciński, "*Random Graphs*", John Wiley & Sons, 2000.

[6] M. Kotowski, M. Kotowski, *Random groups and property (T): Żuk's theorem revisited*, arXiv:1106.2242v2 [math.GR].

[7] Y. Ollivier, *A January 2005 Invitation to Random Groups*, Ensaios Matemáticos [Mathematical Surveys], 10. Sociedade Brasileira de Matemt́ica, Rio de Janeiro, 2005.

[8] Y. Ollivier, *January 2010 random groups updates*, `http://www.yann-ollivier.org/rech/publs/rgupdates.pdf`

[9] A. Żuk, *Property (T) and Kazhdan constants for discrete groups*, Geom. Funct. Anal. 13 (2003), 643–670.

Adam Mickiewicz University, Faculty of Mathematics and Computer Science ul. Umultowska 87, 61-614 Poznań, Poland
E-mail address: `antoniuk@amu.edu.pl`

Adam Mickiewicz University, Faculty of Mathematics and Computer Science ul. Umultowska 87, 61-614 Poznań, Poland
E-mail address: `tomasz@amu.edu.pl`

Instytut Matematyczny, Uniwersytet Wrocławski, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail address: `swiatkow@math.uni.wroc.pl`